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APPROXIMATE SOLUTIONS FOR PERTURBED MEASURE DIFFERENTIAL EQUATIONS WITH MAXIMA

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ABSTRACT

In this paper, we study an approximation of the solutions for perturbed abstract measure differential equations with maxima via hybrid fixed point theorem of Dhage [B.C.Dhage, on some nonlinear alternatives of Leray-Schauder type and functional integral equations, Arch. Math.(Brno) 42(2006), 11-23] under Caratheodory and Lipschitz conditions. An example is also proved to illustrate the hypothesis and abstract theory developed in this paper.

KEYWORDS: The abstract measure differential equation, the abstract measure integral equation, approximation solutions.

INTRODUCTION

The significance of the differential equations with maxima lies in the real world problems of automatic regulation of the technical systems and such differential equations is a special class of functional differential equations in which the state of the unknown function related to the systems depends upon the maximum value of the past state in some past interval of time. The need to study the differential equations with maxima and since then several classes of ordinary and partial differential equations with maxima have been discussed in the literature for different qualitative aspects of the solutions.

Sharma [22 ,23] initiated the study of nonlinear abstract measure differential equations and studied some basic results for such equations. The abstract measure differential equations involve the derivative of the unknown set function with respect to the σ -finite,complete measure. Some of the abstract measure differential equations have been studied by Joshi[17], Shendge and Joshi[24], Dhage [6-9], Dhage and Bellale[11,12], Bellale and Birajdar[4-5] for different aspects of solutions. The Krasnoselskii[18] fixed point theorem is useful for proving approximate solutions for such perturbed differentiions equations under mixed geometrical and topological conditions on the nonlinearities involved in them.

Here our approach is different from that of Sharma [22, 23], The results of this paper complement and generalize the results of the above mentioned papers on abstract measure differential equations under weaker conditions. Therefore, it is of interest to establish algorithms for the hybrid differential equations with maxima for approximation solutions with maxima for approximation solutions along similar lines. The novelty our problem as well as our method is new to the literature in the theory of nonlinear differential equations with maxima.

PRELIMINARIES

A mappings is called D -Lipschitz if there exists a continuous and nondecreasing function $\psi : R^+ \to R^+$ such that

$$\| Tx - Ty \| \le \psi (\| x - y \|) \qquad (2.1)$$

For all $x, y \in X$, where $\psi(0) = 0$. In particular if $\psi(r) = \beta r . \beta > 0$, T is called a Lipschitz with a Lipschitz constant β .

Let X be Banach space and Let $T: X \to X$. T is called compact if $\overline{T(X)}$ is a compact subset of X. T is called totally bounded if for any bounded subset S of X. $T(S)$ is a totally bounded subsets of X. T is called completely continuous if T is continuous and totally bounded on X. Every compact operator is totally bounded, but the converse may not be true, however, two notions are equivalent on bounded subsets of X, The details of different types of nonlinear contraction, compact and completely continuous operators appears in Granas and Dugundji [15].

To prove the main approximation solutions, we need the following nonlinear alternative proved in Dhage [6].

Theorem 2.1: Let U and $\overline{U}$ denote respectively the open and closed bounded subset of a Banach algebra X such that $0 \in U$. Let $A: X \to X$ and $B: \overline{U} \to X$ be two operators such that

(a) A is nonlinear D-contraction, and

(b) B is completely continuous,

Then either

(i) the equation $Ax + Bx = x$ has a solutions in $\overline{U}$, or

(ii) there is a point $u \in \partial U$ such that satisfying $\lambda A\left(\frac{u}{\lambda}\right) + \lambda B u = u$ for some $o < \lambda < 1$, where ∂U is a boundary of U in X.

An interesting corollary to theorem 2.1 in the applicable form

Corollary 2.1: Let $B_r(0)$ and $\overline{B}_r(0)$ denote respectively the open and cloed balls in a Banach algebra X centered at origin 0 of radius r for some real number $r > 0$. Let $A: X \to X$, $B: \overline{B}_r(0) \to X$ be two operators such that

(a) A is contraction, and

(b) B is completely continuous,

Then either

(i) The operator equation $Ax + Bx = x$ has a solution x in X with $\| X \| \le r$, or

(ii) There is an $u \in X$ such that $\|u\| = r$ satisfying $\lambda A\left(\frac{u}{\lambda}\right) + \lambda B u = u$ for some $0 < \lambda < 1$.

In the following section we state our perturbed abstract measure differential equations to discussed qualitively in the subsequent part of this paper.

STATEMENT OF THE PROBLEM

Let X be a real Banach algebra with a convenient norm $\| \, . \, \|$. Let $x, y \in X$. Then the line segment $\overline{xy}$ in X is defined by

$$\overline{xy} = \{ z \in X \mid z = x + r(y - r), \quad 0 \le r \le 1 \}. \tag{3.1}$$

Let $x_0 \in X$ be a fixed point and $z \in X$. Then for any $x \in \overline{x_0 z}$, we define the sets S_x and $\overline{S_x}$ in X by

$$S_x = \{ r\, x \mid -\infty < r < 1 \} \tag{3.2}$$

And

$$\overline{S}_x = \{ r\, x \mid -\infty < r \le 1 \} \tag{3.3}$$

Let $x_1, x_2 \in \overline{xy}$ be arbitrary, we say $x_1 < x_2$ if $S_{x1} \subset S_{x2}$, or equivalently, $\overline{x_0 x_1} \subset \overline{x_0 x_2}$. In this case we also write $x_2 > x_1$ .vector measure (real signed measures) and define a norm $| \, . \, |$ on $ca(X, M)$ by $\| p \| = | p |(X)$

(3.4)

Where $| p |$ a total variation is measure of p and is given by

$$|p|(X) = Sup\sum_{i=1}^{\infty}|p(E_i)|, \qquad E \subset X, \tag{3.5}$$

Where the supremum is taken over all possible partitions $\{E_i : i \in N\}$ of X . it is known that $Ca(X, M)$ is a Banach space with respect to the norm $\|.\|$ given by (3.4)

Let μ be a σ-finite positive measure on X , and let $p \in ca(X, M)$. We say p is absolutely continuous with respect to the measure μ if $\mu(E) = 0$ implies $p(E) = 0$ for some $E \in M$. In this case we also write $p << \mu$.

Let $x_0 \in X$ be fixed and let M_0 denote the $\sigma -$ algebra on S_{x_0} . Let $z \in X$ be such that $z > x_0$ and let M_z denote the σ-algebra of all sets containing M_0 and the sets of the form S_x, $x \in \overline{x_0 z}$.
Given a $p \in ca(X, M)$ with $p << \mu$, consider the abstract measure differential equation(AMDE) of the form

$$\frac{dp}{d\mu} = f(x, p(\overline{S_x})) + g(x, \max_{0 \le \xi \le \overline{S_x}} p(\xi)) \text{ a.e. } [\mu] \text{ on } \overline{x_0 z}. \tag{3.6}$$

And

$$p(E) = q(E), \quad E \in M_0 \tag{3.7}$$

Where q is a given known vector measure, $\frac{dp}{d\mu}$ is a Radon-Nikodym derivative of p with respect to μ, $f, g : S_z \times R \to R$ and $f(x, p(S_x))$ and $g(x, \max_{0 \le \xi \le \overline{S_x}} p(\xi))$ is $\mu -$integrable for each $p \in ca(S_z, M_z)$.

Definition 3.1: Given an initial real measure q on M_0 , vector $p \in ca(S_z, M_z)$ $(z > x_0)$ is said to be a solution of AMDE (3.6)-(3.7) if

(i) $p(E) = q(E)$, $E \in M_0$

(ii) $p << \mu$ On $\overline{x_0 z}$, and

(iii) p Satisfies (3.6) a.e. $[\mu]$ on $\overline{x_0 z}$.

Remark 3.1: The AMDE (3.6) - (3.7) is equivalent to the abstract measure integral equation (in short AMIE)

$$p(E) = \int_E f(x, p(\bar{S}_x))\, d\mu + \int_E g(x, \max_{0 \le \xi \le \bar{S}_x} p(\xi))\, d\mu . \tag{3.8}$$

If $E \in M_z$, $E \subset \overline{x_0 z}$. and

$$p(E) = q(E) \quad \text{if } E \in M_0. \tag{3.9}$$

A solution p of the AMDE (3.6)-(3.7) on $\overline{x_0 z}$ will be denoted by $p(\bar{S}_{x0}, q)$.

APPROXIMATE SOLUTIONS

We need the following definition in what follows.

Definition 4.1: A function $\beta : S_z \times R \to R$ is called Caratheodory if

(i) $x \to \beta(x, y)$ is μ-measurable for each $y \in R$, and

(ii) $y \to \beta(x, y)$ is continuous almost everywhere $[\mu]$ on $\overline{x_0 z}$.

Further a Caratheodory function $\beta(x, y)$ is called L^1_μ-Caratheodory if

(iii) for each real number $r > 0$ there exists a function $h_r \in L^1_\mu(S_z, R)$ such that

$$|\beta(x, y) \leq h_r(x) \quad \text{a.e. } [\mu],\ x \in \overline{x_0\ z}$$

for all $y \in R$ with $|y| \leq r$.

We consider the following set assumptions.

(H_1) For any $z > x_0$, the algebra M_z is compact with respect to the topology generated by the pseudo-metric d defined on M_z by

$$d(E_1, E_2) = |\mu|(E_1 \Delta E_2)$$

for all $E_1, E_2 \in M_z$.

(H_2) $\mu(\{x_0\}) = 0$.

(H_3) There exists a μ-integrable function $\alpha: S_z \to R^+$ such that

$$|f(x, y_1) - f(x, y_2)| \leq \alpha(x)\,|y_1 - y_2| \text{ a.e. } [\mu],\ x \in \overline{x_0\ z}$$

for all $y_1, y_2 \in R$.

(H_4) q is continuous on M_z with respect to the pseudo-metric d defined in (A_0).

(H_5) The function $g(x, y)$ is L^1_μ-Caratheodory.

(H_6) There exists a function $\phi \in L^1_\mu(S_z, R^+)$ such that $\phi(x) > 0$ a.e. $[\mu]$ on $\overline{x_0\ z}$ and a continuous nondecreasing function $\psi: [0, \infty] \to (0, \infty)$ such that

$$|g(x, y)| \leq \phi(x)\,\psi(|y|) \text{ a.e.} [\mu] \text{ on } \overline{x_0\ z}$$

for all $y \in R$.

Theorem 4.1: Suppose that the assumptions $(H_1)-(H_3)$ and $(H_4)-(H_6)$ hold. Further suppose that there exists a real number $r > 0$ such that $\|\alpha\|_{L^1_\mu} < 1$ and

$$r > \frac{F_0 + \|q\| + \|\phi\|_{L^1_\mu}\,\psi(r)}{1 - \|\alpha\|_{L^1_\mu}} \tag{4.1}$$

Where, $F_0 = \int_{\overline{x_0\ z}} |f(x, 0)|\, d\mu$. Then the AMDE $(3.6)-(3.7)$ has a solution on $\overline{x_0\ z}$.

Proof: Consider the open ball $\overline{B}_r(0)$ in $ca(S_z, M_z)$ centered at origin 0 of radius r, where r is a positive real number satisfying the inequality (4.1). Define two operators $A: ca(S_z, M_z) \to ca(S_z, M_z)$, $B: \overline{B}_r(0) \to ca(S_z, M_z)$ by

$$Ap(E) = \begin{cases} \int_E f\left(x, p\left(\overline{S_x}\right)\right) d\mu & \text{if } E \in M_z,\ E \subset \overline{x_0 z} \\ 0, & \text{if } E \in M_0 \end{cases} \tag{4.2}$$

And

$$Bp(E) = \begin{cases} \int_E g\left(x, \max_{0 \leq \xi \leq \overline{S}_x} p(\xi)\right) & \text{if } E \in M_z, \text{if } E \subset \overline{x_0\ z} \\ q(E), & \text{if } E \in M_0 \end{cases} \tag{4.3}$$

We shall show that the operators A and B satisfy all the conditions of Corollary (2.1) on $\overline{B}_r(0)$

Step I: Firstly to show that A is a contraction on $ca(S_z, M_z)$. Let $p_1, p_2 \in ca(S_z, M_z)$ be arbitrary. Then by assumption (A_2)

$$| Ap_1(E) - AP_2(E) | = \left| \int_E f\left(x, p_1(\overline{S}_x)\right) d\mu - \int_E f\left(x, p_2(\overline{S}_x)\right) d\mu \right|$$

$$\leq \int_E \alpha(x) \left| p_1(E) - p_2(E) \right| d\mu$$

$$\leq \| \alpha \|_{L_\mu^1} \, |p_1 - p_2| \, (E)$$

For all $E \in M_z$. Hence by definition of the norm in $ca(S_z, M_z)$ one has

$$\| Ap_1 - Ap_2 \| \leq \| \alpha \|_{L_\mu^1} \| p_1 - p_2 \|$$

For all $p_1, p_2 \in ca(S_z, M_z)$. As a result A is a contraction on $ca(S_z, M_z)$ with the contraction constant $\| \alpha \|_{L_\mu^1}$.

Step II: We show that B is continuos on $\overline{B}_r(0)$. Let $\{p_n\}$ be a sequence of vector measures in $\overline{B}_r(0)$ converging to a vector measure p. Then by dominated convergence theorem,

$$\lim_{n\to\infty} Bp_n(E) = \lim_{n\to\infty} \int_E f\left(x, p_n(\overline{S}_x)\right) d\mu$$

$$= \int_E f\left(x, p(\overline{S}_x)\right) d\mu$$

$$= \overline{B}p(E)$$

For all $E \in M_z$, $E \subset \overline{x_0 z}$ .Similarly, if $E \in M_z$, then

$$\lim_{n\to\infty} \overline{B}p_n(E) = q(E) = Bp(E) .$$

And so B is a continuous operator on $\overline{B}_r(0)$.

Step III: Next we show that B is a totally bounded operator on $\overline{B}_r(0)$. Let $\{p_n\}$be a sequence in $\overline{B}_r(0)$. Then we have $\| p_n \| \leq r$ for all $n \in N$. We shall show that the set$\{ Bp_n : n \in N \}$is a uniformly bounded and equicontinous set in $ca(S_z, M_z)$. In this step first we show that $\{Bp_n\}$is uniformly bounded.

Let $E \in M_z$. Then there exists two subsets $F \in M_0$ and $G \in M_z$, $G \subset \overline{x_0 z}$ such that

$$E = F \cup G \quad \text{and} \quad F \cap G = \phi .$$

Hence by definition of B,

$$|Bp_n(E)| \leq |q(F)| + \int_G \left| g\left(x, \max_{0 \leq \xi \leq \overline{S}_x} p(\xi)\right) \right| d\mu$$

$$\leq \| q \| + \int_G h_r(x) d\mu .$$

$$\leq \| q \| + \int_E h_r(x) d\mu$$

$$= \| q \| + \| h_r \|_{L_\mu^1}$$

From (3.5) it follows that

$$\| Bp_n \| = | BP_n | (S_x)$$

$$= \sup_\sigma \sum_{i=1}^{\infty} | Bp_n(E_i) |$$

$= \| q \| + \| h_r \|_{L^1_\mu}$

For all $n \in N$, Hence the sequence $\{Bp_n\}$ is uniformly bounded in $B(\overline{B}(0))$.

Step IV: To show that $\{ Bp_n : n \in N \}$ is an equicontinuous sequence in $ca(S_z, M_z)$. Let $E_1, E_2 \in M_z$. Then there exist subsets $F_1, F_2 \in M_0$ and $G_1, G_2 \in M_z$, $G_1 \subset \overline{x_0 z}$, $G_2 \subset \overline{x_0 z}$ such way that

$E_1 = F_1 \cup G_1$ with $F_1 \cap G_1 = \phi$ and $E_2 = F_2 \cup G_2$ with $F_2 \cap G_2 = \phi$

We know that,

$$G_1 = (G_1 - G_2) \cup (G_2 \cap G_1), \tag{4.4}$$

And

$$G_2 = (G_2 - G_1) \cup (G_1 \cap G_2), \tag{4.5}$$

Hence ,

$$Bp_n(E_1) - Bp_n(E_2) \le q(F_1) - q(F_2) + \int_{G_1 - G_2} g\left(x, \max_{0 \le \xi \le \overline{S}_x} p(\xi)\right) d\mu + \int_{G_2 - G_1} g\left(x, \max_{0 \le \xi \le \overline{S}_x} p(\xi)\right) d\mu$$

Since $g(x, y)$ is L^1_μ - Caratheodory, We know that,

$$\left| Bp_n(E_1) - Bp_n(E_2) \right| \le | q(F_1) - q(F_2) | + \int_{G_1 \Delta G_2} \left| g\left(x, \max_{0 \le \xi \le \overline{S}_x} p(\xi)\right) \right| d\mu \le | q(F_1) - q(F_2) | + \int_{G_1 \Delta G_2} h_r(x) d\mu$$

.Assume that,

$d(E_1 - E_2) = | \mu | (E_1 \nabla E_2) \to 0.$

Then $E_1 \to E_2$, as a $F_1 \to F_2$ and $|\mu|(G_1 \nabla G_2) \to 0$. As q is continuous on compact M_z .it is uniformly continuous,

$$|Bp_n(E_1) - Bp(E_2)| \le |q(F_1) - q(F_2)| + \int_{G_1 \Delta G_2} h_r(x) d\mu \to 0 \text{ as } E_1 \to E_2.$$

This show that $\{ Bp_n : n \in N \}$ is an equicontinuous set in $ca(S_z, M_z)$. By an application of Arzela-Ascoli theorem yields that B is a totally bounded operator on $\overline{B}_r(0)$. Now B is continuous and totally bounded operator on $\overline{B}_r(0)$. It is completely continuous operator on $\overline{B}_r(0)$. Now an application of Corollary (2.1) holds that either the operator $Ax + Bx = x$ has a solution, or there is a $u \in ca(S_z, M_z)$ with $\|u\| = r$ such that $\lambda A\left(\frac{u}{\lambda}\right) + \lambda Bu = u$ for some $0 < \lambda < 1$. After that we will show that latter assertion does not hold. We will assume contrary, Then

$$u(E) = \begin{cases} \lambda \int_E f\left(x, \dfrac{u(\overline{S}_x)}{\lambda}\right) d\mu + \lambda \int_E g\left(x, \max_{0 \le \xi \le \overline{S}_x} u(\xi)\right) d\mu, & \text{if } E \in M_z, E \subset \overline{x_0 z} \\ \lambda q(E), & \text{if } E \in M_0 \end{cases}$$

for some $0 < \lambda < 1$.

If $E \in M_z$, then there exists sets $F \in M_0$ and $G \in M_z, G \subset \overline{x_0 z}$ such that $E = F \cup G$ and $F \cap G = \phi$. Then we have $u(E) = \lambda A\left(\frac{u(E)}{\lambda}\right) + \lambda Bu(E)$

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$$=\lambda q(F)+\lambda\int_G f\left(x,\frac{u(\bar{S}_x)}{\lambda}\right)d\mu+\lambda\int_G g\left(x,\max_{0\le\xi\le\bar{S}_x}u(\xi)\right)d\mu.$$

Hence,

$$|u(E)|\le\lambda|q(F)|+\int_G\left[\left|f\left(x,\frac{u(\bar{S}_x)}{\lambda}\right)-f(x,0)\right|+|f(x,0)|\right]d\mu+\lambda\int_G\left|g\left(x,\max_{0\le\xi\le\bar{S}_x}u(\xi)\right)\right|d\mu$$

$$\le\lambda\|q\|+\int_G\alpha(x)\left|u(\bar{S}_x)\right|d\mu+F_0+\lambda\int_G\phi(x)\psi\left(\max_{0\le\xi\le\bar{S}_x}u(\xi)\right)d\mu$$

$$\le\|q\|+\|\alpha\|_{L^1_\mu}|u|(E)|+F_0+\int_G\phi(x)\Psi\left(\max_{0\le\xi\le\bar{S}_x}|u|(\xi)\right)d\mu$$

$$\le\|q\|+\|\alpha\|_{L^1_\mu}|u|(E)|+F_0+\int_G\phi(x)\Psi\left(\max\|u\|\right)d\mu$$ ``

$$\le\|q\|+\|\alpha\|_{L^1_\mu}|u|(E)|+F_0+\|\phi\|_{L^1_\mu}\Psi\left(\max\|u\|\right)$$

Which further implies that?

$$\|u\|\le\|q\|+\|\alpha\|_{L^1_\mu}\|u\|+F_0+\|\phi\|_{L^1_\mu}\Psi\left(\max\|u\|\right)$$

Or,

$$\|u\|\le\frac{\|q\|+F_0+\|\phi\|_{L^1_\mu}\psi\left(\max\|u\|\right)}{1-\|\alpha\|_{L\mu^1}}$$

Putting $\|u\|=r=\max\|u\|$ in the equation, we get,

$$r\le\frac{\|q\|+F_0+\|\phi\|_{L^1_\mu}\psi(r)}{1-\|\alpha\|_{L\mu^1}}\qquad(4.6)$$

Which is a contradiction to the inequality in (4.1). In consequence, the operator equation $p(E)=Ap(E)+Bp(E)$ has a solution $u(\bar{S}_{x0},q)$ in $ca(S_z,M_z)$ with $\|u\|\le r$. This further implies that AMDE(3.6)-(3.7). Has a solution on $\overline{x_0z}$. Hence the proof.

4.1 Example:

We consider the following HDE,

$$\frac{dp}{d\mu}=\tan^{-1}p(\bar{S}x)-p(\bar{S}x)+\tanh\left(\max_{0\le\xi\le\bar{S}x}p(\xi)\right)\qquad\mu\ on\ \overline{x_0z}\qquad(4.7)$$

Here $f(\mu,p)=\tan^{-1}p(\bar{S}x)-p(\bar{S}x)$ and $g(\mu,p)=\tanh(\bar{S}x)$

The functions 'f' *and* 'g' are continuous on $\overline{x_0z}$

Next, we have

$$0\le\tan^{-1}p(\bar{S}x_1)-\tan^{-1}p(\bar{S}x_2)\le\frac{1}{\xi^2+1}(\bar{S}x_1-\bar{S}x_2)$$

For all $\bar{S}x_1,\bar{S}x_2$ on $\overline{x_0z}$, $\bar{S}x_1>\xi>\bar{S}x_2$.

Therefore $\lambda = 1 > \frac{1}{\xi^2+1} = \mu$, Hence the function f satisfies the hypothesis (H_3). Moreover, the function $\overline{f}(\mu, p) = \tan^{-1} p(\overline{S}x)$ is bounded on $\overline{x_0 \, z}$ hence hypothesis (H_4) and (H_5) hold. The function $g(\mu, p)$ is increasing in p for each μ *on* $\overline{x_0 \, z}$, hence hypothesis (H_6) is satisfied. Hence HDE (4.7) has a lower solution. Thus all the hypothsis of theorem are satisfied and hence the HDE has a solution

$$\frac{dp}{d\mu} = f(x, p(\overline{S_x})) + g(x, \max_{0 \le \xi \le \overline{S}_x} p(\xi)) \quad \text{a.e.}[\mu] \text{ on } \overline{x_0 \, z}$$

and

$$p(E) = q(E), \quad E \in M_0.$$

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